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Notes on $p^{\omega+1}$-Projective Abelian p-Groups

by

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Abstract. We prove that for any $n \in \mathbb{N} \cup \{0\}$ the $p^{\omega+n}$-projective abelian p-groups with first Ulm factor which is a direct sum of pure-complete groups are direct sums of countable groups of lengths no more than $\omega + n$. This refines a result due Irwin-Keef (J. Algebra, 1993). In particular, the $p^{\omega+n}$-projective pure-complete abelian p-groups are precisely the direct sums of cyclic p-groups. This is a strengthening of a result due to P. Hill (Glasgow Math. J., 1972).

We also show that the $p^{\omega+1}$-projective abelian Σ-p-groups (in particular the $p^{\omega+1}$-projective summable abelian p-groups) are direct sums of countable p-groups each of which has length at most $\omega + 1$. This enlarges in some aspect a result of B. Wick (Pac. J. Math., 1984). We also give an example that this is not true for all other $p^{\omega+n}$-projectives with $2 \leq n < \omega$.

Certain related assertions are established as well.

Throughout this brief article, suppose that G is a p-torsion abelian group with first *Ulm* subgroup denoted as $G^1 = p^\omega G = \bigcap_{i<\omega} p^i G$. For every natural number $n \geq 1$ the subgroup $G[p^n] = \{g \in G | p^n g = 0\}$ is called the p^n*-socle* of G, and any subgroup S of $G[p^n]$ is called a p^n*-subsocle* of G; when $n = 1$ they are termed as *socle* and *subsocle*, respectively. The (p^n)-socles and (p^n)-subsocles play an important role in the theory of abelian groups (see e.g. [12], [13]). For instance, it is well-known by Fuchs [15] that the $p^{\omega+n}$-projective p-groups, defined in [25] by the usage of homological machinery, for $n \in \mathbb{N}$ may be classified up to isomorphism via their valuated p^n-socles (see [16] when $n = 1$). Fuchs also asked whether $p^{\omega+1}$-projective groups could be characterized in terms of filtrations (= continuous ascending chains of subgroups terminating at G). Every $p^{\omega+1}$-projective group possesses a filtration whereas, unfortunately, in [11] was demonstrated via a concrete counterexample that the converse implication is impossible. The $p^{\omega+n}$-projective groups can also be identified in a purely algebraic sense. Specifically, the following excellent criterion of Nunke holds fulfilled.

THEOREM ([25]). *Let G be an abelian p-group. Then G is $p^{\omega+n}$-projective* $\Leftrightarrow \exists$ $C \leq G[p^n]$: *G/C is a direct sum of cyclic groups.*

Thereby, at first look, the $p^{\omega+n}$-projectives are not too far removed from direct sums of cyclics; note that the $p^{\omega+n}$-projective groups are *fully starred* ([19], p. 446, Corollary

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2). However, in [9], Dieudonné has proven the remarkable fact that there exists a separable $p^{\omega+1}$-projective p-group which is not a direct sum of cyclic groups; so the $p^{\omega+n}$-projective groups, even being separable, need not be a direct sum of countable groups although the separable direct sums of countable groups are known to be direct sums of cyclics (see [26] or [14]).

Thus, it would be rather natural to ask the following question.

QUESTION: Under what additional circumstances on the group structure every $p^{\omega+n}$-projective group is a direct sum of countable groups?

The objective of the present short paper is to show that this is realizable via the existence of quite popular conditions.

All unexplained exclusively notation and notions can be found essentially in [14].

First of all, before presenting the main attainments, we list the following technicality; it is an analogue of a result by Kulikov (e.g. [14, p. 143, Theorem 28.2]) but is not its immediate consequence.

PROPOSITION ([1, Theorem 2.3]). *Let G be a p-primary abelian group and suppose that there exists a pure subgroup K of G such that $G/K[p^n]$ is a direct sum of cyclic groups. Then G is a direct sum of cyclic groups.*

REMARK. In the case when $n = 1$, the last statement is Theorem 1 or Corollary 7 from [20].

An abelian p-group G is said to be *pure-complete* if each subsocle of G supports a pure subgroup of G.

So, we have at our disposal all the information necessary for proving the following.

PROPOSITION 1. *Suppose G is an abelian p-group. Then G is $p^{\omega+1}$-projective so that G/G^1 is pure-complete if and only if G is a direct sum of countable groups of lengths not exceeding $\omega+1$. In particular, if G is separable then G is a direct sum of cyclic groups.*

Proof. Foremost, assume that G is separable, that is $G^1 = 0$. Thus, by the text, G is both $p^{\omega+1}$-projective and pure-complete. Owing to the foregoing criterion of Nunke, there is $C \leq G[p]$ with the property that G/C is a direct sum of cyclics. On the other hand, since G is pure-complete, there is a pure subgroup K of G so that $C = K[p]$. That is why $G/K[p]$ is a direct sum of cyclics and we obviously observe that the quoted above Proposition is applicable to get that G is a direct sum of cyclic groups.

Now we treat the general case. We shall show first that G being $p^{\omega+1}$-projective forces that so is G/G^1 (see [26, p. 194, Corollary 2.4] too). In doing this, by using the Nunke's criterion we write that G/C is a direct sum of cyclics where $pC = 0$, hence $G^1 \subseteq C$. Certainly, $p(C/G^1) = 0$ and $G/G^1/C/G^1 \cong G/C$ is a direct sum of cyclic groups. Consequently, again with the aid of Nunke's criterion we are done.

Furthermore, by what we have shown in the first situation, G/G^1 is a direct sum of cyclics. But $p^{\omega+1}G = 0$, whence G^1 is bounded. Therefore, invoking [26] (see [14] as well), we conclude that G is indeed a direct sum of countable groups, as asserted.

The converse implication is routine since in view of [26] (see also [14]) it well-known that the direct sums of countable groups of length $\leq \omega + 1$ are $p^{\omega+1}$-projective as well as they have first Ulm factor that is a direct sum of cyclic groups. QED

An abelian p-group G is said to be $p^{\omega+1}$-*injective* whenever $p^{\omega+1}G = 0$ and G/G^1 is torsion-complete. Moreover, it is long-known that each torsion-complete group is pure-complete.

As direct consequences, we derive the following.

COROLLARY 1. *The abelian p-group G is* $p^{\omega+1}$*-projective and* $p^{\omega+1}$*-injective only when G is a bounded p-group.*

Proof. Since G/G^1 is torsion-complete, the previous Theorem 1 implies that G/G^1 is a torsion-complete direct sum of cyclics. Consequently, according to [14], this quotient is bounded. Because G is reduced, it easily follows that it must also be bounded, as stated. QED

Consider now the direct sum of torsion-complete groups $A = \bigoplus_{i \in I} A_i$. If $|I| < \aleph_0$, then A is torsion-complete; if $|I| = \aleph_0$, then A is pure-complete (see, for example, [14, p. 48, Lemma 73.1]); however if $|I| > \aleph_0$, then A is not necessarily pure-complete.

And so, we are ready to generalize the last corollary to the next

COROLLARY 2. *Suppose G is an abelian p-group. Then G is* $p^{\omega+1}$*-projective such that* G/G^1 *is a direct sum of torsion-complete groups precisely when G is a direct sum of countable groups of lengths less than or equal to* $\omega + 1$. *In particular, G is a direct sum of cyclic groups provided that it is separable.*

Proof. As we have already seen in Proposition 1, G/G^1 is $p^{\omega+1}$-projective, whence all summands of this factor-group are one also (see [26]). Therefore, by the usage of Proposition 1 or the preceding Corollary 1, it must be that G/G^1 is a direct sum of cyclics. Furthermore, the proof goes on again in virtue of the same procedure as in Theorem 1.QED

Another non-trivial direct consequence is the following one; we give a somewhat more conceptual proof.

COROLLARY 3 ([17]). *Suppose that G is a subgroup of a direct sum of Pruefer p-groups. If G is pure-complete, then G is a direct sum of cyclic groups.*

Proof. What suffices to argue is that any subgroup of a direct sum of Pruefer p-groups is $p^{\omega+1}$-projective. In order to do that, let $G \subseteq P$ where P is a direct sum of Pruefer p-groups. Letting $S = G \cap P^1$, we obtain that $pS = 0$ since $p^{\omega+1}P = 0$ and also that $G/S \cong (G + P^1)/P^1 \subseteq P/P^1$ is a direct sum of cyclics as being a subgroup of a direct sum of cyclics (cf. [14]). Consequently, referring to alluded to above Nunke's criterion, we have that G is really $p^{\omega+1}$-projective. Henceforth, Proposition 1 works. QED

The results obtained in Proposition 1 and Corollary 2 can be summarized in the next

THEOREM 1. *Suppose G is an abelian p-group. Then G is $p^{\omega+1}$-projective such that G/G^1 is a direct sum of pure-complete groups if and only if G is of length not exceeding $\omega + 1$ a direct sum of countable groups. In particular, G is a direct sum of cyclic groups when it is separable.*

Proof. It relies on the same idea as in Corollary 2. QED

REMARK. It is long-known by Hill (e.g. [17]) that even the finite direct sum of pure-complete groups need not be pure-complete, so Theorem 1 is a proper generalization of Proposition 1 as well as of Corollary 2 since there exists a pure-complete group which is not torsion-complete.

Irwin and Keef proved in [18] that if G is $p^{\omega+n}$-projective for some $n < \omega$ and is pure-complete, then G is a direct sum of cyclics. Using this result, our Theorem 1 can be extended to $p^{\omega+n}$-projectives for any $n \geq 1$; however we have followed a different approach which works for other sorts of groups as well–for example such as the direct sums of thick groups which shall be examined below.

Next, we concern other major class of primary abelian groups which properly contains the torsion-complete ones ([2], Corollary 3.4). The members of such a class are called *thick groups*. For freely use in the sequel, we state the following affirmation.

CRITERION. *The abelian p-group G is thick $\Leftrightarrow$ there exists a pure subgroup K of G such that G/K is bounded and $K[p] = H[p]$ for every subgroup H of G so that G/H is a direct sum of cyclic groups.*

Proof. It follows by a subsequent combining of Theorems 3.2 point (3) and 1.3 points (4) and (3) from [2]. QED

It is also worthwhile noticing that we have argued in [9] that G is thick if and only if G/G^1 is thick, whenever G is an abelian p-group. In particular, if G is a direct sum of thick groups then it plainly follows that so does G/G^1. The opposite implication is unknown yet.

So, we have accumulated all the information seemingly needed for proving the following.

THEOREM 2. *Suppose G is an abelian p-group. Then G is $p^{\omega+1}$-projective such that G/G^1 is a direct sum of thick groups if and only if G is a direct sum of countable groups of length no more than $\omega + 1$. In particular, if G is separable then G is a direct sum of cyclic groups; and if G is separable thick then G is bounded.*

Proof. First of all, let we presume that G is separable thick. Since G is $p^{\omega+1}$-projective, an appeal to the Nunke's necessary and sufficient condition leads us to the existence of $C \leq G[p]$ so that G/C is a direct sum of cyclics. Knowing this, with the aid of the foregoing criterion for thickness, we yield that there is a pure subgroup K of G with the property that $G/K[p] = G/C$ is a direct sum of cyclic groups. With this in hand, we may exploit the above Proposition from [1] to infer that G is a direct sum of cyclics. But G being thick does not have unbounded direct summands of cyclic groups. Finally, we deduce that G is bounded.

Further, for the general situation, the proof goes on in the same way as in Theorem 1 and Corollary 2, supported by the preliminary commentaries. QED

REMARK. The preceding statement is another not elementary extension of Corollary 2. It may also be deduced by exploiting Corollary 25 of [18]. In [4] and [5], Cutler and Cutler-Missel have constructed a C-indecomposable $p^{\omega+n}$-projective p-group for $n \geq 2$, that is an essentially finitely indecomposable $p^{\omega+n}$-projective group but which is not C-decomposable. Using the methods proposed there, we can extend our Theorem 3, listed below, to C-decomposable $p^{\omega+n}$-projective p-groups. Note once again that each $p^{\omega+1}$-projective p-group is C-decomposable (cf. [16]).

An abelian p-group G is named *essentially finitely indecomposable* if it has no an unbounded direct sum of cyclic summands; notice the helpful fact that the thick groups are obviously essentially finitely indecomposable (cf. [2]). There exists a conjecture due to J. M. Irwin (e.g. [2]) that the essentially finitely indecomposable groups are precisely the thick ones. This is still left-open and seems to be in the distant future. Moreover, another interesting for the current exploration fact due again to Irwin (see Corollary 3.3 of [2]) is that the pure-complete groups are essentially finitely indecomposable if and only if they are thick.

We are now prepared to prove the following improvement of Theorem 2.

THEOREM 3. *Suppose G is an abelian p-group. Then G is $p^{\omega+1}$-projective such that G/G^1 is a direct sum of essentially finitely indecomposable groups if and only if G is a direct sum of countable groups of length at most $\omega+1$. In particular, if G is separable then G is a direct sum of cyclic groups; and if G is separable essentially finitely indecomposable then G is bounded.*

Proof. First of all, let G be separable essentially finitely indecomposable $p^{\omega+1}$-projective. Since G is both C-decomposable ([16], Corollary 3) and essentially finitely indecomposable, it is straightforward that it must be bounded. But a direct summand of an essentially finitely indecomposable group obviously belongs to this group class, henceforth in the general situation the proof relies on the same argumentation as in Theorem 1 and Corollary 2. QED

We close the statements of the above type presented with the class of so-called Σ-groups. An abelian group G is said to be a *Σ-group* if some its high subgroup is a direct sum of cyclic groups. It is well-known that each totally projective p-group is a Σ-group as well as each summable p-group is a Σ-group, and that the separable p-torsion Σ-groups are precisely the direct sums of p-cyclics.

The following two necessary and sufficient conditions are of usefulness.

CRITERION ([6]). *The abelian p-group G is a Σ-group $\Leftrightarrow$ $G[p]=\bigcup_{i<\omega} G_i$, $G_i \subseteq G_{i+1} \leq G$ and, $\forall\, i \geq 1$, $G_i \cap p^i G = G^1[p]$.*

In [8] (see [7] too) we have generalized the classical criterion of Kulikov for direct sums of cyclic groups and its fundamental expansion of Dieudonné ([10]) by showing that they are equivalent, in fact. In [7] we demonstrate that a classical theorem of Fuchs and Mostowski-Sosiada ([14], p. 111, Proposition 18.3) is decidable from the result of Dieudonné [10].

Now, for application in the sequel, we pose the following global version of the Dieudonné's criterion, that version is equivalent to the classical one and the proof of which is similar to the methods illustrated in [8].

CRITERION ([10]; [8]). *Let A be an abelian p-group with a subgroup C so that A/C is a direct sum of cyclic groups. Then A is a direct sum of cyclic groups* $\Leftrightarrow C = \bigcup_{i<\omega} T_i, T_i \subseteq T_{i+1}$ *and* $\forall\ i \geq 1 : T_i \cap p^i A = 0 \Leftrightarrow C[p] = \bigcup_{i<\omega} C_i, C_i \subseteq C_{i+1}$ *and* $\forall\ i \geq 1 : C_i \cap p^i A = 0$.

We indicate that there exists by Megibben [22] a Σ-p-group G so that G/G^1 is unbounded torsion-complete. Such a group G is obviously not a direct sum of countable groups. Inspired by Theorem 1, we have even more that the group G is not $p^{\omega+1}$-projective. This shows once again the well-known fact that the classes of Σ-p-groups and $p^{\omega+1}$-projective p-groups are absolutely different. Knowing this, it is rather natural to asks when they coincide, or, in more precise words, what is their intersection.

We also remember the classical result due to Rangaswamy [28] who had established that every cotorsion (in particular reduced algebraically compact) Σ-group is bounded; thus every algebraically compact Σ-group is the direct sum of a divisible group and a bounded group.

So, we are in a position to proceed by proving the following.

THEOREM 4. *Suppose G is an abelian p-group. Then G is a $p^{\omega+1}$-projective Σ-group if and only if G is a direct sum of countable groups of length less than or equal to $\omega + 1$.*

Proof. Referring to the stated above criterion of Nunke from [25], we write that there exists $C \leq G[p]$ so that G/C is a direct sum of cyclics; thus $G^1 \subseteq C$ and $G/G^1/C/G^1 \cong G/C$ is a direct sum of cyclic groups. On the other hand, since $C \subseteq G[p]$, an appeal to the early formulated group criterion from [6] allows us to derive that $C = \bigcup_{m<\omega} C_m$, where $C_m = C \cap G_m \subseteq C \cap G_{m+1} = C_{m+1}$ and, for all natural numbers, we compute that $C_m \cap p^m G = C \cap (G_m \cap p^m G) = C \cap G^1[p] \subseteq G^1[p]$. Therefore, $C/G^1 = (C/G^1)[p] = \bigcup_{m<\omega}[(C_m + G^1)/G^1]$, and applying the modular law we calculate that $[(C_m + G^1)/G^1] \cap p^m(G/G^1) = [(C_m + G^1)/G^1] \cap (p^m G/G^1) = [(C_m + G^1) \cap p^m G]/G^1 = (G^1 + (C_m \cap p^m G))/G^1 = G^1/G^1 = \{0\}$. We now see that all the conditions in the already listed above global version of the classical criterion of Dieudonné are satisfied, hence we deduce that G/G^1 is a direct sum of cyclics. Finally, because $p^n G^1 = 0$, we take into account [26] (see [14] as well) to conclude that G is really a direct sum of countable groups, as wanted. QED

As just noted, since every summable p-group is known to be a Σ-group, in conjunction with the preceding Theorem 3 we obtain the following. Actually, these two assertions are equivalent since we have proved in [6] that the classes of Σ-groups and summable groups both of length which does not exceed $\omega + n$ (n is an arbitrary non-negative integer) do coincide.

COROLLARY 4. *Suppose that G is an abelian p-group. Then G is a $p^{\omega+1}$-projective summable p-group if and only if G is of length not exceeding $\omega+1$ a direct sum of countable groups.*

REMARK. In [29] was proven that an S-group is p^{α}-projective for some ordinal number α if and only if it is totally projective of length less than or equal to α. It is well-known that any S-group, being a special isotype subgroup of a totally projective group, is a Σ-group. Thereby, our Theorem 4 (and Corollary 4, respectively) may be considered as an extension of the cited fact.

We want to show another independent confirmation of Theorem 4, whence of Corollary 4 too, like this. In ([16], p. 446, Corollary 3), Fuchs and Irwin showed that each $p^{\omega+1}$-projective abelian p-group A may be decomposed as follows: $A = B \oplus C$, where B is separable $p^{\omega+1}$-projective and C is a direct sum of countable p-groups of lengths no more than $\omega + 1$. Furthermore, if we choose A to be a Σ-group as well, its summand C being pure in A is also a Σ-group. But it is separable, hence as earlier noticing C is a direct sum of cyclics and the result that A must be of length $\leq \omega + 1$ a direct sum of countable p-primary groups now follows.

Nevertheless, their decomposable result was established only for $p^{\omega+1}$-projective groups because the idea in [16] is based on the properties of the socle as a valuated vector space. We also remark that in ([16], p. 468, Theorem 6) was proved that any $p^{\omega+1}$-projective abelian p-group can be embedded by a special manner in a direct sum of countable p-groups of lengths less than or equal to $\omega + 1$. Since for $n \geq 2$ the p^n-socle $A[p^n]$ is no longer a valuated vector space, we claim that such a decomposition property does not hold for any $p^{\omega+n}$-projective groups with $n > 1$ (see [5]). However, if A is C-decomposable, A can be decomposed in such a way as indicated, namely if A is a C-decomposable $p^{\omega+n}$-projective p-group then in [5] was proved that $A = B \oplus C$ where B is a C-decomposable $p^{\omega+n}$-projective p-group with $p^{\omega+n-1}B = 0$ and C is a direct sum of countable p-groups of length at most $\omega + n$. Notice once again that all $p^{\omega+1}$-projective groups are C-decomposable (cf. [16]), while for $p^{\omega+n}$-projectives with $n \geq 2$ this fails to be true.

We come now to the showing that Theorem 4 is not valid for $p^{\omega+n}$-projectives when $n > 1$, even provided that they are C-decomposable. Specifically, the following well-known classical counterexample holds.

EXAMPLE. Let A be a separable p-group which is $p^{\omega+1}$-projective but not a direct sum of cyclics (e.g. [10]), and let B be a basic subgroup of A. Put $G = A/B[p]$. Then it can be checked that the following properties hold:

(1) $p^{\omega}G = A[p]/B[p]$;

(2) $H = B/B[p] \cong pB$ is a high subgroup of G which subgroup is a direct sum of cyclics, so G is a Σ-group;

(3) $G/p^{\omega}G \cong A/A[p] \cong pA$ is not a direct sum of cyclics, therefore G is not a direct sum of countables;

(4) G is proper $p^{\omega+2}$-projective of length $\omega + 1$.

Proof of (1), (2), (3) *and* (4).

(1) It is easily seen that $A[p] = (p^i A)[p] + B[p]$, $\forall\ i < \omega$, hence $A[p] = \bigcap_{i<\omega}((p^i A)[p] + B[p]) \subseteq \bigcap_{i<\omega}(p^i A + B[p])$ and thus $A[p]/B[p] \subseteq p^{\omega}G$.

On the other hand $p^{\omega}G = \bigcap_{i<\omega} p^i G = \bigcap_{i<\omega}(p^i A + B[p])/B[p] \subseteq ((pA)[p] + B[p])/B[p] = A[p]/B[p]$ since $p(\cap_{i<\omega}(p^i A + B[p])) = \bigcap_{i<\omega}(p^i A) = p^{\omega}A = 0$, and thus the reverse inclusion $p^{\omega}G \subseteq A[p]/B[p]$ is true. Whence the desired equality.

(2) Utilizing the previous point, it is a straightforward computation to show that $(B/B[p]) \cap p^{\omega}(A/B[p]) = (B/B[p]) \cap (A[p]/B[p]) = (B \cap A[p])/B[p] = B[p]/B[p] = 0$. Moreover, by the same token, for another $C \leq A$ so that $B \subseteq C$ we have that $(C/B[p]) \cap p^{\omega}(A/B[p]) = C[p]/B[p] = 0$. That is why, $C[p] = B[p]$. But B being pure in A is pure in C as well. Furthermore, we wish apply a simple technical lemma from [14] to conclude that $C = B$ as required.

(3) That $G/p^{\omega}G \cong A/A[p] \cong pA$ is not a direct sum of cyclics follows from the aforementioned result due to Fuchs-Mostowski-Sosiada (see, for instance, [14]) since otherwise A must be a direct sum of cyclic groups which contradicts our choice. Thereby, the fact that G is not a direct sum of countable groups follows by conforming with [26] or [14].

(4) We claim that G is $p^{\omega+2}$-projective but not $p^{\omega+1}$-projective. Indeed, since in virtue of (3) the factor-group $G/p^{\omega}G$ is $p^{\omega+1}$-projective being imbedded in the $p^{\omega+1}$-projective group A, there exists $K \leq G$ such that $p^{\omega}G \subseteq K$ and $K/p^{\omega}G \subseteq (G/p^{\omega}G)[p]$ with G/K a direct sum of cyclics. But $pK \subseteq p^{\omega}G$, hence $p^2K \subseteq p^{\omega+1}G = p(A[p]/B[p]) = 0$. Consequently, the Nunke's criterion enables us to infer that G is really $p^{\omega+2}$-projective. Since $p^{\omega+1}G = 0$ while by (1) $p^{\omega}G = A[p]/B[p] \neq 0$ (otherwise $A[p] = B[p]$ implies that $A = B$ is a direct sum of cyclic groups which is false owing to our hypothesis), we derive that $length(G) = \omega + 1$.

Next, assume in a way of contradiction that G is $p^{\omega+1}$-projective. Therefore, property (2) along with Theorem 4 guarantee that G is a direct sum of countable groups. Thus, according to [26] or [14], $G/p^{\omega}G$ should be a direct sum cyclic groups which is against property (3). Another reason that G is not $p^{\omega+1}$-projective, the reader can see in ([21, p. 4383]). Even more, there G may be chosen to be C-decomposable.

We mention that the example can be extended to any $p^{\omega+n}$-projectives for $n \geq 2$ by replacing $G = A/B[p]$ with $G = A/B[p^{n-1}]$.

The preceding decomposition of $p^{\omega+1}$-projective groups ([16], Corollary 3) is convenient for the consideration of a class of groups which is much more large than the class of Σ-groups. Namely, an abelian p-group G is called a *highly pure-complete group* or a *highly essentially finitely indecomposable group* if its high subgroups are pure-complete groups or essentially finitely indecomposable groups, respectively.

The following assertion expands in some aspect Theorems 1 and 3. Unfortunately, our result will be restricted only for $p^{\omega+n}$-projective groups with $n = 1$; the general case for $n > 1$ is wrong as the above counterexample has illustrated.

THEOREM 5. *Suppose G is an abelian p-group. If G is both $p^{\omega+1}$-projective and either a highly pure-complete group or a highly essentially finitely indecomposable group, then G is a direct sum of countable groups of length not exceeding $\omega + 1$.*

Proof. By virtue of ([16], Corollary 3) we may write $G = B \oplus C$, where B is separable $p^{\omega+1}$-projective and C is direct sum of countable groups with $length(C) \leq \omega+1$. Letting H_C be an arbitrary high subgroup of C, we derive that $B \oplus H_C$ is high in G. Hence B as being a direct summand is a pure-complete group or an essentially finitely indecomposable group. Consequently, Theorem 1 or Theorem 3 is applicable to infer that B is a direct sum of cyclics. That is why, G is a direct sum of countable groups, and we are done. QED

REMARK. The same method can be applied to abelian groups whose high subgroups are direct sums of pure-complete groups or direct sums of essentially finitely indecomposable groups (in particular of thick groups), if the direct summands of such direct sums inherit the corresponding property. Moreover, by what we have argued above, each abelian p-group which high subgroup is $p^{\omega+1}$-projective and either pure-complete or a direct sum of essentially finitely indecomposable groups is a Σ-group.

About the next concepts we refer the reader for more details to [27]. We shall consider two sorts of abelian p-groups. Firstly, we deal with the reduced *strongly straight p-groups* that are of necessity separable. They properly contain the *semi-complete p-groups*, and also are contained in the pure-complete ones. That is why it is of great interest to handle the more large class of *straight p-groups*. It contains the $p^{\omega+1}$-injective p-groups and, instead of the class of strongly straight groups, there exists a separable straight p-group which is not pure-complete. We emphasize that the direct sum of Pruefer p-groups, which is of course a direct sum of countable groups of length that does not exceed $\omega + 1$, is both $p^{\omega+1}$-projective and straight.

Besides, a separable abelian p-group G is named a *Q-group* provided that $|(G/H)^1| \leq |H|$ whenever $H \leq G$ with $|H| \geq \aleph_0$ (cf. [19]).

We terminate the current work with some questions.

PROBLEMS: Firstly, does it follow that the theorems remain true for the classes of $p^{\omega+1}$-projective p-groups and (1) almost totally injective p-groups; (2) (separable) straight p-groups? Notice that these two classes properly include the class of torsion-complete p-groups.

We note also that in [23] (see [24] too) it is shown that it is consistent that an $\aleph_1$-separable p-group of cardinality $\aleph_1$ is $p^{\omega+n}$-projective for each $n \geq 1$ if and only if it is n-pseudo-free. Since only 0-pseudo-free groups are precisely the direct sums of cyclics, we observe that the above type theorems does not hold for $\aleph_1$-separable groups (whence not for Q-groups because they properly encompass the $\aleph_1$-separable ones) even when $n = 1$. It is well-known by Eklof-Mekler that, assuming Martin Axioms plus the denial of the Continuum Hypothesis, every $\aleph_1$-separable p-group of final rank $\aleph_1$ is C-decomposable. We remark as well that there is a $p^{\omega+1}$-projective p-group of cardinality $\aleph_1$, with the additional set-theoretical assumption $2^{\aleph_0} < 2^{\aleph_1}$, that is a Q-group but not an $\aleph_1$-separable group (see [13, Corollary 3.6]).

Secondly, as above emphasized, in [4] and [5] was constructed an essentially indecomposable $p^{\omega+2}$-projective p-group which is not C-decomposable, hence is not a direct sum of cyclics.

Thirdly, under what extra limitation on the group structure, for an arbitrary ordinal α the p^{α}-projective Σ-p-group (in particular the p^{α}-projective summable p-group with $\alpha \leq \omega_1$) is totally projective of length $\leq \alpha$? In this aspect, as already noted, it is well-known by Wick [29] that an S-group is p^{α}-projective for some ordinal α only if it is totally projective with length not exceeding α; recall that each S-group is a Σ-group. Thereby, the problem can equivalently be reformulated thus: under what circumstances the Σ-groups are S-groups?

REMARK. In ([7], Corollary 10) we have proved in details that the abelian p-group G is pure-complete if and only if $p^n G$ is pure-complete for some positive integer n. Irwin and Swanek used in the proof of Corollary 9 from [20] this equivalence of the property pure-completeness as they refer the readers to [3] for results relating $p^n G$ and G (see, for instance, the proof of Theorem 1 in [20]). But, throughout the rest, in [3] there is no result of such a type for pure-complete groups.

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